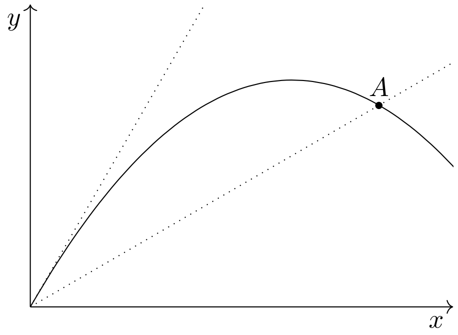
4901. Sketch the curve $x + \sqrt{3}y = \sin\left(\sqrt{3}x - y\right)$.

4902. A projectile is launched from an origin at speed u. The angle of projection is set so that the projectile requires the minimum launch speed to hit point A.



Prove that the direction of projection is the angle bisector of the vertical and vector $\overrightarrow{OA}$.

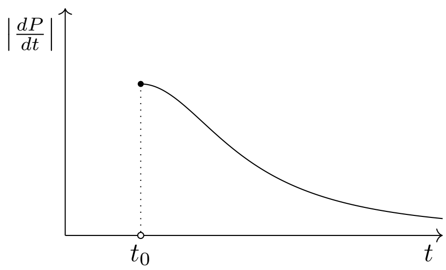
4903. Determine the simultaneous solution of

$$y = x^4 - 2x^2,$$
$$x = y^4 - 2y^2.$$

4904. An applied mathematician is modelling the way in which a crowd disperses after a big event. He uses a continuous variable P to describe the number of thousands of people present at an open-air concert, and t as time in minutes. He models the rate at which people leave as follows, using constants N, k and t_0.

$$\left|\frac{dP}{dt}\right| = \begin{cases} 0, & t < t_0, \\ \dfrac{N}{1 + k(t - t_0)^2}, & t \geq t_0. \end{cases}$$

The positive quantity $\left|\frac{dP}{dt}\right|$ represents the rate at which the number present decreases, i.e. the rate at which people leave. The graph of the above is



(a) Interpret the constants t_0 and k.

(b) Show that, if the total crowd is P_0 thousands,

$$N = \frac{2P_0\sqrt{k}}{\pi}.$$

(c) Sketch a graph of P against t.

4905. Prove the identity

$$\frac{\sin 3\theta}{\cos\theta} + \frac{\cos 3\theta}{\sin\theta} \equiv 2\cot 2\theta.$$

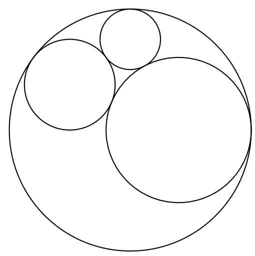
4906. Expressions E_1, E_2, E_3 are defined as

$$E_1 = x^2 - 1,$$
$$E_2 = \left(x + \sqrt{3}y\right)^2 - 4,$$
$$E_3 = \left(x - \sqrt{3}y\right)^2 - 4.$$

Sketch the inequality $E_1E_2E_3 < 0$.

4907. Prove that, if a polynomial $\mathrm{f}(x)$ and all of its derivatives up to and including the nth are zero at $x = p$, then $\mathrm{f}(x)$ has a factor of $(x - p)^{n+1}$.

4908. Three circles are placed inside a fourth circle, such that every pair of circles is tangent. In descending order, the circles have radii $1, \frac{3}{5}, \frac{3}{8}, r$.



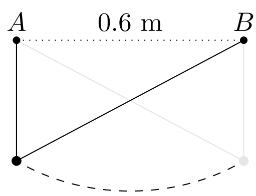
Find the length r of the shortest radius.

4909. An equation is given, for some constant $n \in \mathbb{N}$, as

$$\sum_{r=1}^{n} \frac{(-1)^{r+1}}{(x - r)^2} = 0.$$

Show that this equation has at least $n - 1$ roots. (In fact, it has exactly $n - 1$ roots, but you needn't show this.)

4910. A heavy ring of mass m is threaded onto a smooth, light, inextensible string. The string is 1 m long, and its ends are attached to two points A and B, which are 0.6 m apart horizontally. The ring is released from rest in the position shown, vertically below A. It moves periodically along the dashed path shown, with the string taut throughout.



(a) Verify that the trajectory of the ring may be described by part of the ellipse

$$16x^2 + 25y^2 = 4.$$

(b) Find the set of possible values for the direction of the velocity of the ring. Your answer should be a subset of $[0, 2\pi)$.

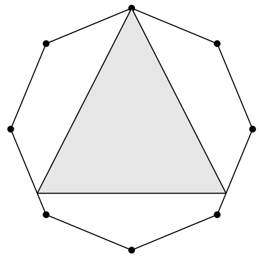
4911. Find the following integral

$$\int \sin(\ln x)\,dx.$$

4912. Find $a, b \in \mathbb{Z}$ such that

$$a + b\sqrt{2} = \sqrt[3]{38752 + 28310\sqrt{2}}.$$

4913. The diagram shows an isosceles triangle inscribed symmetrically in a regular octagon, with its apex at one of the vertices.

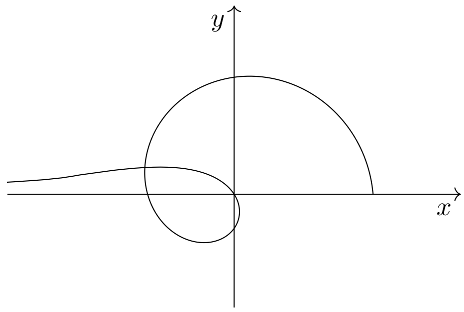


Show that the area of the triangle is maximised if its other two vertices coincide with those of the octagon.

4914. The *radius of curvature* is defined as the radius of the circular arc which best approximates a curve at a particular point. For the curve $y = \cos x$, find the radius of curvature at $x = \pi$.

4915. For $t \in [0, 2\pi)$, a parametric graph is given by

$$x = \ln t \cdot \cos t,$$
$$y = -\ln t \cdot \sin t.$$



The curve has a point of self-intersection. Show that the parameters s, t at this point are given by

$$s, t = \frac{\sqrt{\pi^2 + 4} \mp \pi}{2}.$$

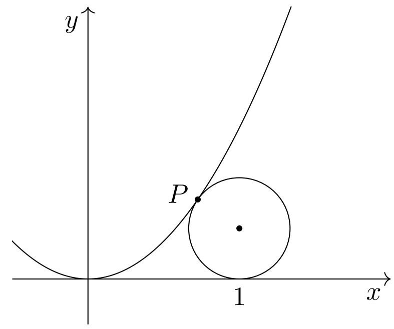
4916. In this question, $a, b \in \mathbb{R}$ are constants with $a \neq b$. A quartic function h has the following properties:

$$\mathrm{h}(a) = \mathrm{h}(b),$$
$$\mathrm{h}'(a) + \mathrm{h}'(b) = 0,$$
$$\mathrm{h}''(a) = \mathrm{h}''(b) = 0.$$

Show that, for all $x \in \mathbb{R}$,

$$\mathrm{h}\left(\frac{a+b}{2} - x\right) = \mathrm{h}\left(\frac{a+b}{2} + x\right).$$

4917. The circle $(x-1)^2 + (y-a)^2 = a^2$, where $a \in \mathbb{R}^+$, and the parabola $y = x^2$ are tangent to each other at a point P, with $x = p$.



Determine p, to 4sf.

4918. Show that, on the unit circle, the average value of the expression x^2y^2 is $\frac{1}{8}$.

4919. In this question, use the result

$$\frac{d}{dx}(x \ln x - x) \equiv \ln x.$$

Consider the following integral:

$$I = \int x^{n-1} \ln x\,dx.$$

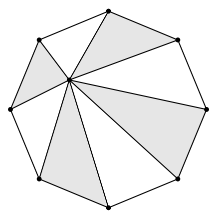
(a) Show that $I = \dfrac{1}{n^2} \displaystyle\int n x^{n-1} \ln\big(x^n\big)\,dx$.

(b) Hence, show that $I = \dfrac{x^n}{n^2}(n \ln x - 1) + c$.

4920. Show that, if a normal is drawn to $y = x^2$, then any re-intersection with the curve satisfies $y \geq 2$.

4921. Rationalise the denominator of $\dfrac{1}{1 + \sqrt[4]{2}}$.

4922. From a point in the interior of a regular $2n$-gon, triangles are constructed to each edge. Every other triangle is shaded, as in the example $n = 4$ below.



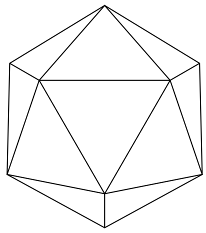
Prove that half of the polygon is shaded.

4923. A hand of four cards is dealt from a standard deck. Show that the average number of suits present is approximately 2.785.

4924. Sketch the locus of points satisfying the following implicit relation:

$$x^3 - x^2y^2 = xy - y^3.$$

4925. An icosahedron has 20 equilateral triangular faces. Five faces meet at each vertex.



A design for an icosahedral die has the numbers 1 to 20 printed on these faces at random.

Find the probability that the numbers 1 to 5 surround one vertex.

4926. A circle is given by $(x-d)^2 + (y-r)^2 = r^2$, for $0 < r < d$. Two tangents are drawn to the circle, each of which passes through the origin.

(a) Sketch the situation, verifying that one of the tangents is the x axis.

(b) Write down the equation of the line L_1 which passes through the origin and the centre of the circle.

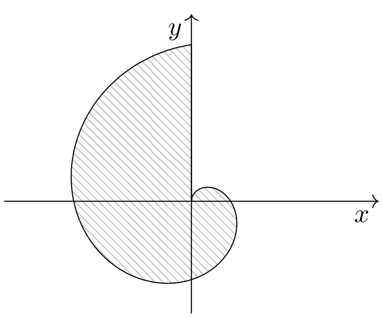
(c) Find the equation of the line L_2 which passes through $(d, 0)$ and is normal to L_1.

(d) Determine algebraically the coordinates of the intersections of line L_2 and the circle.

(e) Hence, prove that the lengths of the tangents from a point to a circle are the same.

4927. The Archimedean spiral shown below is defined, for $t \in [0, 2\pi]$, by the parametric equations

$$x = t\sin t,$$
$$y = t\cos t.$$



(a) Show that the area shaded is given by

$$A = \tfrac{1}{2}\int_0^{2\pi} t^2 + t^2\cos 2t + t\sin 2t \, dt.$$

(b) Hence, determine the exact value of A.

4928. A loop of smooth, light string is pulled taut, into an isosceles triangle, by three coplanar forces of magnitude A, A and B, where $2A > B$.

Prove that the tension in the string is given by

$$T = \frac{A^2}{\sqrt{4A^2 - B^2}}.$$

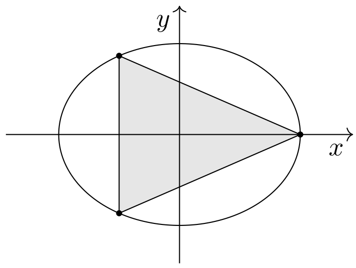
4929. Let n be a natural number, and let $\mathrm{g}(n)$ be the number of distinct positive integer divisors of n, including 1 and n.

For example, 6 has four distinct positive integer divisors $\{1, 2, 3, 6\}$, so $\mathrm{g}(6) = 4$.

Prove that $\mathrm{g}(n)$ is odd iff n is a perfect square.

4930. An isosceles triangle, with a vertex at $(a, 0)$ and the x axis as its line of symmetry, is inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Show that the area of the triangle satisfies

$$A_\triangle \leq \tfrac{3\sqrt{3}}{4}ab.$$

4931. Sketch the following graphs, where $a < b$,

(a) $x + y = (x - y - a)(x - y - b)$,

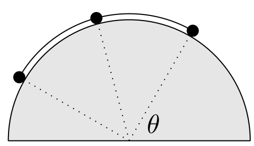
(b) $x + y = (x - y - a)^2(x - y - b)$.

4932. The triangle numbers are given by

$$T_n = \tfrac{1}{2}n(n+1).$$

The sum, from $n = 1$ to infinity, of the reciprocals of the triangular numbers converges. Prove that it converges to 2.

4933. Three particles with masses m, m, $2m$ are lying in equilibrium on the smooth surface of a half-cylinder of radius r. They are attached to each other by light, inextensible strings, each of length $\frac{1}{4}\pi r$. The radius to the heavier mass is at angle of inclination θ.

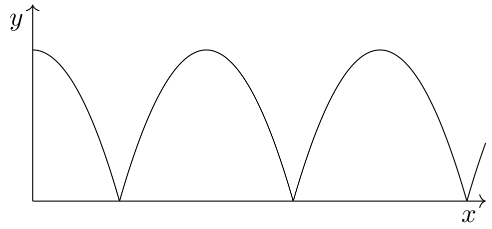


Show that $\tan\theta = 3 - \sqrt{2}$.

4934. For a constant $k \in (0, 1)$, shade the region

$$\left(x^2 + y^2 - 1\right)\left((x + k - 1)^2 + y^2 - k^2\right) \leq 0.$$

4935. A projectile is launched horizontally from the point $(0, c)$ with speed u ms^{-1}. It bounces, without loss of speed, on the ground at $y = 0$. It continues in this manner, bouncing repeatedly.



Show that the equation of the trajectory between the n^{th} and $(n + 1)^{\text{th}}$ bounces is given by

$$y = -\frac{gx^2}{2u^2} + \frac{nx}{u}\sqrt{8cg} - c\left(4n^2 - 1\right).$$

4936. (a) Show that

$$\int_0^{\frac{\pi}{2}} \frac{4\cos x}{(1 + \cos x)^3}\, dx = \int_0^{\frac{\pi}{4}} 2\sec^4 t - \sec^6 t\, dt.$$

(b) Hence, determine the value of

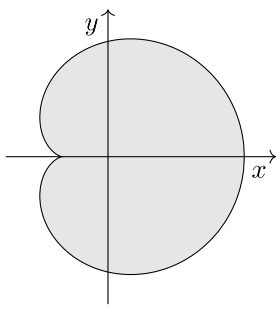
$$\int_0^{\frac{\pi}{2}} \frac{5\cos x}{(1 + \cos x)^3}\, dx.$$

4937. A *perfect number* is a positive integer which is equal to the sum of all its positive divisors except itself. Prove an ancient result of Euclid: if $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is a perfect number.

4938. Show that, if $xy\,(x + y) = 16$, then $x^2 + y^2 \geq 8$.

4939. A *cardioid* is defined, for $0 \leq \theta < 2\pi$, by

$$\begin{aligned} x &= 4\cos\theta + 2\cos 2\theta, \\ y &= 4\sin\theta + 2\sin 2\theta. \end{aligned}$$



Determine the area enclosed by the curve.

4940. Show that the following order-10 polynomial has exactly two real roots

$$x^{10} - 2x^8 + x^6 - x^4 + 2x^2 - 2 = 0.$$

4941. A straight line $y = mx + c$ is tangent to the curve $y = x^3 - x^4$ at two distinct points. Find c.

4942. This question concerns a limit:

$$L = \lim_{x \to 0^+} \sin(\ln x).$$

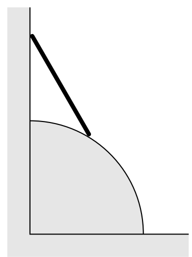
The notation $x \to 0^+$ means that x tends to zero from above, i.e. x is positive and tends to zero.

(a) Show that L is not $\pm\infty$.

(b) Show that L is not 0.

(c) Hence, explain why L is not well defined.

4943. A precarious ladder of length 1 is set up as shown below, standing on the rough surface of a fixed quarter-cylinder of radius 1 and leaning against a smooth vertical wall. The ladder makes an angle α with the vertical, and the coefficient of friction at the contact with the quarter-cylinder is μ. The system is in equilibrium.



Show that $\mu \geq \dfrac{3\sin 2\alpha}{3\cos 2\alpha + 1}$.

4944. The *Euler substitution* is

$$\sqrt{x^2 + k} = -x + t.$$

Using this substitution, show that

$$\int \frac{1}{\sqrt{x^2 + k}}\, dx = \ln\left|x + \sqrt{x^2 + k}\right| + c.$$

4945. With sin taking inputs in radians, set S is

$$\left\{x \in \mathbb{Q} : \sin\left(x^2\right) > {}^1\!/_2\right\}.$$

Prove that S has infinitely many elements.

4946. A plane is coloured such that every point is either red, green or blue. Prove that there must be two points of the same colour which are 1 unit apart.

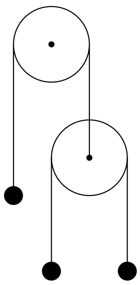
4947. True or false:

(a) A quintic in x must have a real root $x = \alpha$,

(b) A quintic in x^2 must have a real root $x = \alpha$,

(c) A quintic in x^3 must have a real root $x = \alpha$.

4948. A pulley system is set up as follows. A string is passed over a fixed pulley. A 4 kg mass is attached to one end of the string, and a movable pulley to the other. A string runs over this second pulley, with 2 kg and 3 kg masses attached to its ends. The system is released from rest.



(a) Explain the physical assumption(s) necessary to conclude the following:

i. the tension is the same throughout each of the strings,

ii. the tension in one string is twice as great as the tension in the other.

(b) Making these assumptions, show that the 4 kg mass accelerates at $\frac{1}{11}g$ ms^{-2}.

4949. Four events A, B, C, D have equal probabilities $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = \mathbb{P}(D)$, and no more than two of them can happen together. Determine the set of possible values of

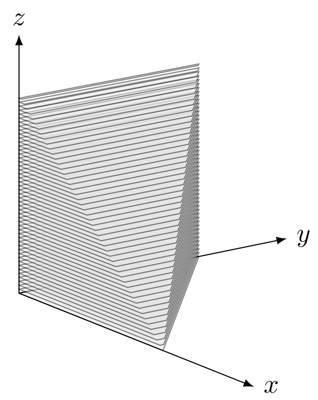
(a) $\mathbb{P}(A \cap B)$,

(b) $\mathbb{P}(A \cup B \cup C)$.

4950. Prove the following identity:

$$\sin\left(\tfrac{1}{2}\arcsin x\right) \equiv \tfrac{1}{2}\left(\sqrt{1+x} - \sqrt{1-x}\right).$$

4951. A tetrahedral sculpture is being 3D printed. Its height varies from $z = 0$ to $z = 1$. When at height $z = k$, the printer is instructed to fill, in the (x, y) plane, the region bounded by the lines $x + y = 1$, $x = 0$, $y = 0$ and $x = 1 - k$.



By setting up and evaluating a definite integral, determine the volume of the sculpture.

4952. A firework propels particles at every angle to the vertical, at speed u, in an (x, y) plane. Each of the particles sparkles when its reaches its maximum height. Show that the sparkles form an ellipse, and give its Cartesian equation.

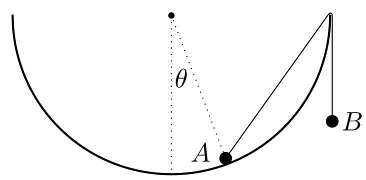
4953. On an n-by-n grid, where $n \geq 5$, $n - 2$ counters are to be placed, with a maximum of one per grid square. Let p be the probability that the counters are collinear. Prove that

$$p = \frac{n(n^2+3)}{{}^{n^2}\mathrm{C}_{n-2}}.$$

4954. The graph $y = \mathrm{f}(x)$ is rotated by 45° anticlockwise around the origin. Determine the equation of the transformed graph.

4955. *This problem was contributed by Sukumar Chandra to the Feynman Lectures on Physics website.*

Particle A, of mass $6m$, sits in equilibrium on the smooth inner surface of a hemispherical bowl. This is shown in cross-section. A light string connects particle A to particle B, of mass m, hanging freely outside the bowl. Between them, the string rests on the lip of the bowl, which is also smooth. At the centre of the bowl, the radius to A subtends an angle θ with the vertical.



(a) Show that $6\sqrt{2}\sin\theta = \cos\frac{1}{2}\theta + \sin\frac{1}{2}\theta$.

(b) Hence, or otherwise, show that $\theta = \arcsin\frac{1}{8}$.

4956. An integral is defined as $I = \displaystyle\int \sec^3 x$.

(a) Verify, by differentiation, that

$$\int \sec x = \ln|\sec x + \tan x| + c.$$

(b) Use integration by parts to show that

$$I = \tfrac{1}{2}\big(\sec x \tan x + \ln|\sec x + \tan x|\big) + d.$$

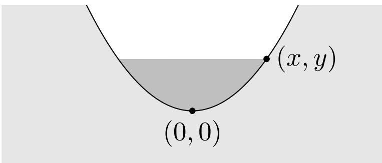
4957. Prove that, if $2^k + 1$ is a prime number greater than 2, then k is a power of 2.

4958. A surface is defined by the equation

$$z = \sin(x+y) + \sin(x-y).$$

The surface is reflected in the plane $z = 1$, then translated by the vector $\pi\mathbf{i} + 2\pi\mathbf{j} - 2\mathbf{k}$. Find and simplify the equation of the new surface.

4959. A drainage ditch is modelled as having constant cross-section $y = \frac{3}{4}x^2$, with $(0,0)$ the deepest point. The cross-sectional area of the water, which is shaded darker below, is denoted A.



Water flows in constantly, at a rate of 3 m^3 per metre of ditch per month. Water is also lost by evaporation: per month, the rate at which volume is lost by this means is given by

$$\frac{dV}{dt} = -1.5S,$$

where S is the surface area of the water.

(a) Show that
  i. $A = x^3$,
  ii. $\dfrac{dA}{dt} = 3(1-x)$.

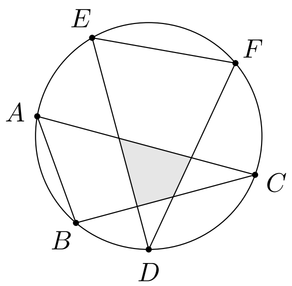
(b) Show that, whatever the initial conditions, the depth tends to 75 cm.

4960. A student chooses a specific function g, and then writes down the following differential equation:

$$\frac{d^2y}{dx^2} - \mathrm{g}(x) = 0.$$

Having solved the equation, she notices that, if two distinct curves $y = \mathrm{f}_1(x)$ and $y = \mathrm{f}_2(x)$ both satisfy the DE, then they intersect a maximum of once. Determine whether, if she chose a different function g, she would see the same effect.

4961. Six points A, B, C, D, E, F are chosen at random on the circumference of a circle. Triangles ABC and DEF are drawn.



Find the probability that, as in the outcome shown above, there is a region common to both triangles which is quadrilateral.

4962. A plane P and a sphere S are defined by

$$P : x + y + z = 9,$$
$$S : x^2 + y^2 + z^2 = 9.$$

Find the shortest distance between P and S.

4963. The nth term of a sequence is given, in terms of the golden ratio $\phi = \frac{1}{2}(1+\sqrt{5})$, by

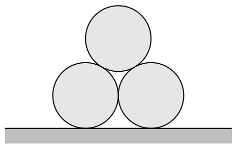
$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}.$$

By showing that $F_{n+2} = F_{n+1} + F_n$, prove that F_n generates the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$

4964. By considering the RHS as the number of ways of choosing a committee of $(r+1)$ people from a group of $(n+1)$, or otherwise, prove that

$$\sum_{i=r}^{n} {}^{i}\mathrm{C}_r \equiv {}^{n+1}\mathrm{C}_{r+1}.$$

4965. Cylindrical ice cores, each of radius r and mass m, are stacked and held on a horizontal surface. This is shown below in cross-section. All contacts are modelled as smooth.



They are released from rest. Show that the upper core begins accelerating downwards at $\frac{1}{7}g$ ms^{-2}.

4966. The *Weierstrass substitution* uses the change of variables $t = \tan\frac{x}{2}$. Show that, using it,

$$\int \mathrm{f}(\sin x, \cos x)\ dx$$

can be written in the form

$$\int \mathrm{f}\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2\,dt}{1+t^2}.$$

4967. This question is about an iteration similar to the Newton-Raphson iteration, in which $y = \mathrm{f}(x)$ is approximated by a parabola, rather than a line.

(a) Determine, in terms of $\mathrm{f}(k)$, $\mathrm{f}'(k)$ and $\mathrm{f}''(k)$, the coefficients a, b, c of the quadratic graph $y = ax^2 + bx + c$ that most closely follows the curve $y = \mathrm{f}(x)$ at $x = k$.

(b) For $\mathrm{f}(x) = \ln x - x$, show that
  i. $a = -\dfrac{1}{2k^2}$,
  ii. $b = \dfrac{2}{k} + 1$
  iii. $c = \ln k - \frac{3}{2}$.

(c) The equation $\ln x + x = 0$ has one root, at around 0.567143. With starting value $k = 0.5$, show that the next approximation is given by

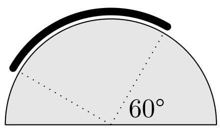
$$\frac{5 - \sqrt{21 - 8\ln 2}}{2}.$$

4968. A curve has parametric equations

$$x = t - t^3,$$
$$y = t + t^3.$$

Show that the curve is inflected at the origin.

4969. A uniform rope of mass m and length $\frac{1}{2}\pi$ is placed on the smooth surface of a half-cylinder of radius 1. The radius to one of its ends is inclined at 60° to the horizontal.



Show that the initial acceleration of the rope is

$$a = \frac{\sqrt{3} - 1}{\pi} g \text{ ms}^{-2}.$$

4970. A monic cubic graph $y = \text{f}(x)$, whose coefficients are integers, has a tangent drawn to it at $x = a$, where $a \in \mathbb{Z}$. This tangent re-intersects the curve at $\big(b, \text{f}(b)\big)$. Prove that b is an integer.

4971. An ellipse has defining equation

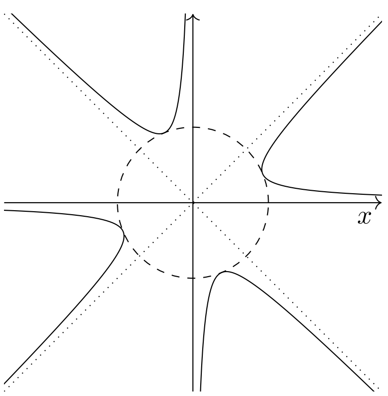
$$(y - x)^2 + y^2 = 1.$$

Show that the lines of symmetry of the ellipse are

$$2y = \big(-1 \pm \sqrt{5}\big)x.$$

4972. Prove that, if $2^p - 1$ is prime, then p is prime.

4973. The diagram shows a curve, the circle $x^2 + y^2 = 4$ and the lines $y = \pm x$. The curve is asymptotic to the lines $y = \pm x$, and to the axes.

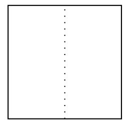


The curve has the form

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 = 4.$$

Find the constants a, b, c, d, e.

4974. Three circles are to be placed inside a unit square, with their radii and positions chosen to maximise the total area. The arrangement is to be set up, without overlap between the circles, such that there is a line of symmetry as shown:



Find, with careful proof, the fraction of the area of the square covered in the optimal arrangement.

4975. This problem is known as *Buffon's needle*.

A needle of length 1 is dropped at random, both in orientation and position, onto floorboards of width 1. Prove that the probability that the needle will lie across one of the cracks is $p = {}^2/_\pi$.

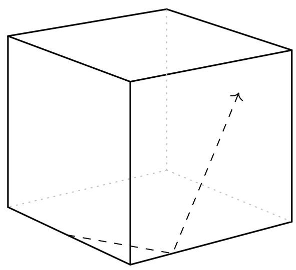
4976. This question concerns the *double tangents* of a quintic graph Q, where we define a double tangent to be a straight line which is tangent to a curve at two distinct points. The curve Q is given by

$$y = x^5 - 20x^3 + 64x.$$

(a) Show by sketching that there are exactly three double tangents to Q.

(b) Find the gradients of the double tangents.

4977. An ant is walking on a unit cube. It starts at the midpoint of an edge, and sets off in a straight line, at an angle $\arctan \frac{1}{2}$ to that edge. It continues straight, even when crossing an edge, such that its path between two points on adjacent faces is always the shortest one.



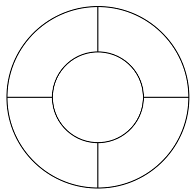
This behaviour produces periodic motion, with the ant repeatedly walking a loop. Find the length of the loop.

4978. The equation $x + y + z = 1$ defines a plane in 3D. In the positive octant $x, y, z \geq 0$, it encloses a tetrahedral region T. Prove, without quoting a formula, that the volume of T is ${}^1/_6$.

4979. Sketch the region(s) of the (x, y) plane satisfying

$$x^4y^2 - x^2y^4 - x^2 + y^2 \leq 0.$$

4980. The regions of the following diagram are randomly coloured, each red, yellow, green or blue. Multiple regions can be the same colour.



Find the probability that no two regions sharing a border end up coloured the same.

4981. Find the following indefinite integral:

$$\int \sin 5x \sin 3x \, dx.$$

4982. A skier is skiing down a mountainside. Skis are designed to slide forwards easily, while resisting sliding side to side. In a simplified model for this, the coefficients of friction between skis and snow are modelled as different in different directions:

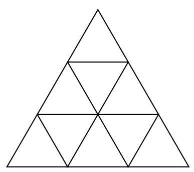
① μ_1 parallel to the skis,
② μ_2 perpendicular to them.

The mountainside is modelled as a plane inclined at θ above the horizontal. These values satisfy

$$0 < \mu_1 < \tan\theta < \mu_2.$$

The skier sets off on the steepest straight course for which constant speed is possible. Show that the angle ϕ between the skis and the line of greatest slope satisfies $\cos\phi = \mu_1 \cot\theta$.

4983. A pattern consisting of nine small triangles is as depicted below. It is to be coloured with three colours, such that no two small triangles sharing an edge are to be the same colour.



Determine the number of possible colourings.

4984. Prove that, if $k \in \mathbb{Z}$ and $\sqrt{k} \notin \mathbb{Z}$, then $\sqrt{k} \notin \mathbb{Q}$.

4985. A circle in the (x, y) plane moves, parametrised by time t, according to

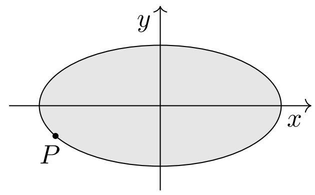
$$(x - 2\sin 2t)^2 + (y - 2\sin t)^2 = 1.$$

Prove that every point (x, y) for which $x^2 + y^2 \le 4$ lies on the circle at some time t.

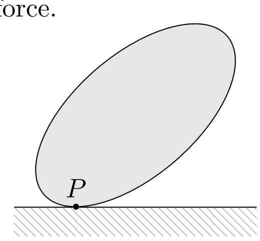
4986. A smooth, uniform, elliptical prism of mass m kg has a cross-section bounded, in units of metres, by

$$x = \sqrt{6}\cos t,$$
$$y = \sqrt{2}\sin t.$$

Point P is located at $t = \frac{7\pi}{6}$.



The prism is now to be positioned with point P in contact with the ground. To achieve equilibrium, an anticlockwise *couple* is applied, which exerts a turning effect in the plane of the cross-section, but no resultant force.



Find the magnitude of the applied couple.

4987. Sketch the solution set of the inequality

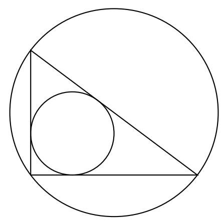
$$x^2 + |y| \le y^2 + |x|.$$

4988. A square domain S is defined with both x and y in the interval $[-2, 2]$. A point (x, y) is chosen at random inside S. Show that

$$\mathbb{P}\left(x^{10000}y + xy^{10000} \ge 1\right) \approx \tfrac{3}{8}.$$

4989. Prove that every cubic graph may be transformed, via a combination of stretches, translations and reflections, onto exactly one of the cubics $y = x^3$, $y = x^3 - x$ or $y = x^3 + x$.

4990. On a right-angled triangle ABC, two circles are drawn: the *incircle* tangent to all three sides and the *circumcircle* through all three vertices.



Prove that the radii r and R of these circles satisfy

$$rR = \frac{abc}{2(a+b+c)}.$$

You may wish to use the half-angle formula

$$\tan\tfrac{1}{2}\theta \equiv \frac{\sin\theta}{1 - \cos\theta}.$$

4991. (a) Use the substitution $u = \sec\theta + \tan\theta$ to find

$$\int \sec\theta \, d\theta.$$

(b) Use the substitution $x = \sec\theta$ and integration by parts to determine a simplified expression for the following integral:

$$I = \int \frac{2x^2}{\sqrt{x^2-1}} \, dx.$$

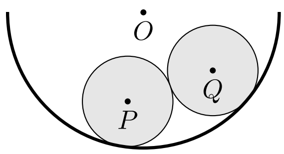
4992. A constant $k > 0$ is such that

$$\begin{aligned} & x^2 + y^2 = k \\ \implies & (x^2y^2 - 1)\left((x^2 - y^2)^2 - 4\right) \leq 0. \end{aligned}$$

Determine the value of k.

4993. Prove the following special case of *Bézout's lemma*: if the highest common factor of $a, b \in \mathbb{N}$ is 1, then there exist $x, y \in \mathbb{Z}$ such that $ax + by = 1$.

4994. Two spheres of radius r are in equilibrium inside a smooth hemispherical bowl of radius $3r$. The masses of the spheres are m and km, where $k > 1$. Points O, P and Q are the centres of, respectively, the bowl and the heavier and lighter spheres. You may assume that the combined centre of mass of the spheres divides PQ in the ratio $1 : k$.



Show that the angle θ between OP and the vertical is given by $\cot\theta = k\sqrt{3}$.

4995. Two curves C_1 and C_2 are defined as $y = x^2$ and its image under rotation by angle θ clockwise around the origin. Prove that the area A of the region enclosed by the curves is given by

$$A = \frac{\sin^3\theta}{3(1-\cos\theta)^3}.$$

4996. An n-tuple of numbers is chosen, each randomly on the interval $[0, 1]$. Determine the probability p_n that these numbers sum to less than 1.

4997. *Brahmagupta's formula* states that the area of a cyclic quadrilateral, whose sides have lengths p, q, r, s, is given, in terms of the semiperimeter $S = \frac{1}{2}(p + q + r + s)$, by

$$A = \sqrt{(S-p)(S-q)(S-r)(S-s)}.$$

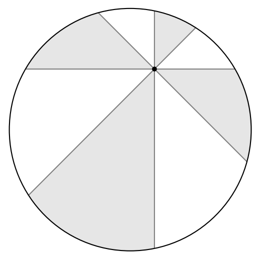
Prove Brahmagupta's formula.

4998. Let A_n, for $n \in \mathbb{N}$, be the area of the region of the (x, y) plane which satisfies both of the following:

$$\begin{aligned} (x+y)^{2n} - (x-y) &< 1, \\ (x-y)^{2n} - (x+y) &< 1. \end{aligned}$$

Prove rigorously that $\lim_{n\to\infty} A_n = 2$.

4999. The diagram below shows a circle and a point (a, b) inside the circle. Chords through (a, b) parallel to $x = 0$, $y = 0$ and $y = \pm x$ divide the interior of the circle into eight sections. This is the setup for the *pizza theorem* with four chords.



Prove that half of the circle is shaded.

5000. Three curves are given by

$$\begin{aligned} 0 &= y - x^2, \\ 0 &= 6 - 2\sqrt{5} - (x+3)^2 - y^2, \\ 0 &= 27\left(1+\sqrt{5}\right)(y+1) + 4x^4 + 32x^3 + 72x^2. \end{aligned}$$

Prove that the curves are equidistant.

——— End of Volume V ———